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HOLOMORPHIC IMMERSIONS OF A COMPACT KÄHLER MANIFOLD INTO COMPLEX TORI

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In this paper we shall study the holomorphic immersion of a compact connected Kähler manifold M into a complex torus. It is easily seen that M admits a holomorphic immersion into a cmoplex torus if and only if the holomorphic cotangent bundle $T^*(M)$ is ample (see § 1), and it has been proved in a joint paper [6] by the author with W. Stoll that if $T^*(M)$ is ample and one of the Chern numbers of M is nonzero, then M is algebraic.

This paper is devoted mainly to the study of a compact connected n-dimensional Kähler manifold M admitting a holomorphic immersion Φ into an $(n+1)$-dimensional complex torus B. The image $X = \Phi(M)$ defines a positive irreducible divisor $D(X)$ and a holomorphic line bundle which we shall denote by $\{X\}$. The Chern class $c(\{X\})$ of $\{X\}$ is represented by a unique $(1,1)$-form of the type $\frac{1}{2}i \sum h_{kj}\zeta^k \wedge \bar{\zeta}^j$, where $\{\zeta^1, \cdots, \zeta^{n+1}\}$ is a basis of the space of holomorphic 1-forms on B, and $H_\Phi = (h_{kj})$ is a constant Hermitian matrix. It is known from the theory of theta functions that the Hermitian matrix H_Φ is positive (Weil [9]). Our main purpose is to describe the properties of M in term of the Hermitian form H_Φ, and our main results are as follows.

First we show that we can reduce the case, where H_Φ is degenerate, to the case where H_Φ is positive definite. Namely let $\mathrm{Aut}_0(M)$ be the identity component of the group of holomorphic transformations of M. It is well-known that the group $\mathrm{Aut}_0(M)$ is a complex Lie group, and in our special case we prove that $\mathrm{Aut}_0(M)$ is a complex torus acting freely on M and the complex dimension of $\mathrm{Aut}_0(M)$ is equal to the nullity of the Hermitian form H_Φ. Then M is a holomorphic principal bundle over the quotient $N = M/\mathrm{Aut}_0(M)$ with structure group $\mathrm{Aut}_0(M)$. The quotient manifold N is algebraic and admits a holomorphic immersion Ψ into a complex torus whose complex dimension is $\dim N + 1$, and the Hermitian matrix H_Ψ associated with the immersion Ψ is positive definite. We shall show that the following conditions are equivalent: 1) H_Φ is positive definite, 2) $\dim \mathrm{Aut}_0(M) = 0$, 3) the Euler number $E(M)$ of M is nonzero.

We then prove that a compact connected n-dimensional Kähler manifold M with $E(M) \neq 0$ admits a holomorphic immersion into an $(n+1)$-dimensional complex torus if and only if the holomorphic cotangent bundle of M

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is ample and M has precisely $n+1$ linearly independent holomorphic 1-forms. Furthermore, the Albanese variety A of M is $(n+1)$-dimensional, the canonical map $J: M \rightarrow A$ is an immersion, and any holomorphic immersion Φ of M into an $(n+1)$-dimensional complex torus B is obtained from the canonical map J composing with a homomorphism of A onto B and a translation of B.

It is shown that every divisor in M is a divisor of a "theta function" on the universal covering manifold of M, and we shall obtain expressions of the Euler number, the arithmetic genus and the plurigenera of M in terms of the elementary divisors of the imaginary part of the Hermitian matrix H_J associated with the canonical map $J: M \rightarrow A$ and the "degree" of J.

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1. Let $B = C^m/\varDelta$ be a complex torus of dimension m, where $\varDelta$ is a lattice in C^m. We shall denote by π the canonical projection of C^m onto B. Let $\{w^1, \cdots, w^m\}$ be the standard coordinates in C^m. Then 1-forms dw^k are invariant by translations and hence projectable onto B. There exist therefore m linearly independent holomorphic 1-forms $\zeta^1, \cdots, \zeta^m$ on B such that

$$\pi^*\zeta^k = dw^k \qquad (k = 1, \cdots, m) ,$$

and these 1-forms are invariant by translations in B and form a basis of all holomorphic 1-forms in B. We shall denote by T_b the translation of B by an element $b \in B$. We identify the holomorphic tangent vector space $T_y(B)$ of B at each point $y \in B$ with C^m by the identification map $T_y(B) \rightarrow C^m$ which assigns to each vector $u \in T_y(B)$ the m-dimensional vector $(\zeta^1(u), \cdots, \zeta^m(u)) \in C^m$.

Let M be an n-dimensional complex manifold, and

$$\Phi : M \rightarrow B$$

be a holomorphic map from M into B, and let

$$\omega^k = \Phi^*\zeta^k \qquad (k = 1, \cdots, m) . \tag{1.1}$$

Then $\omega^1, \cdots, \omega^m$ are holomorphic 1-forms on M and the differential $\Phi_*(x)$: $T_x(M) \rightarrow T_{\Phi(y)}(B) = C^m$ is given by

$$\Phi_*(x)(u) = (\omega^1(u), \cdots, \omega^m(u)) , \qquad u \in T_x(M) . \tag{1.2}$$

Let x_0 be the reference point of M chosen once for all. Then

$$\Phi(x) = T_{\Phi(x_0)}\left(\pi\left(\int_{x_0}^x \omega^1, \cdots, \int_{x_0}^x \omega^m\right)\right) , \tag{1.3}$$

where $\int_{x_0}^{x} \omega^k$ means the integral of ω^k along a path γ from x_0 to x, and it can be shown that $\pi\left(\int_{x_0}^{x} \omega^1, \cdots, \int_{x_0}^{x} \omega^m\right)$ does not depend on the choice of γ.

Let Φ_1 and Φ_2 be two homotopic holomorphic maps from M to B, and assume that M is compact. Since Φ_1 and Φ_2 are homotopic, closed holomorphic forms $\Phi_1^*\zeta^k$ and $\Phi_2^*\zeta^k$ have the same periods and are therefore cohomologous. Thus $\Phi_1^*\zeta^k - \Phi_2^*\zeta^k = df$ and $df = d'f, d''f = 0$, so that f is holomorphic and hence a constant, and we get $\Phi_1^*\zeta^k = \Phi_2^*\zeta^k$. It follows then from (1.3) that

$$\Phi_2(x) = T_b(\Phi_1(x)) , \qquad b = \Phi_2(x_0)\cdot\Phi_1(x_0)^{-1} .$$

From now on we always assume that M is compact and connected, and let Aut (M) be the group of all holomorphic transformations of M. It is well-known that Aut (M) has a structure of complex Lie group such that Aut $(M) \times M \to M$ defined by $(g, x) \to g\cdot x$ is a holomorphic map. We shall denote by Aut_0 (M) the identity component of Aut (M). If $g \in \mathrm{Aut}_0$ (M), then g is homotopic to the identity map of M, and hence Φ and $\Phi \circ g$ are homotopic for any holomorphic map Φ from M to B. Therefore

$$\Phi(gx) = T_{\Phi'(g)}(\Phi(x)) ,$$

where

$$\Phi'(g) = \Phi(gx_0) \circ \Phi(x_0)^{-1} .$$

It is easily seen that Φ' is a homomorphism of the complex Lie group Aut_0 (M) into the complex torus B. The Lie algebra $\mathfrak{G}$ of all holomorphic vector fields on M is identified with the Lie algebra of Aut_0 (M). The Lie algebra of B is also identified with the Lie algebra of all translation invariant vector fields X of type (1,0) which is identified with C^m by the map $X \to (\zeta^1(X), \cdots, \zeta^m(X))$. Then the homomorphism of Lie algebra $\Phi'_*: \mathfrak{G} \to C^m$ induced by Φ': Aut_0 $(M) \to B$ is given by

$$\Phi'_*(\xi) = (\omega^1(\xi), \cdots, \omega^n(\xi)) , \tag{1.4}$$

where $\xi \in \mathfrak{G}$ is a holomorphic vector field on M.

Let M be a compact connected Kähler manifold, $\{\theta^1, \cdots, \theta^q\}$ $(q = h^{1,0}(M))$ a basis of the vector space of all holomorphic 1-forms on M, and $H_1(M, Z)$ the 1-dimensional integral cohomology group of M. Then the image Δ_A of the homomorphism $\gamma \to \left(\int_\gamma \theta^1, \cdots, \int_\gamma \theta^q\right)$ of $H_1(M, Z)$ into C^q is a lattice of C^q, and the complex torus

$$A = C^q/\Delta_A$$

is called the Albanese variety of M. The map $J: M \to A$ defined by

$$J(x) = \pi\left(\int_{x_0}^{x} \theta^1, \cdots, \int_{x_0}^{x} \theta^q\right)$$

is holomorphic, and $J^*(\zeta^k) = \theta^k$ $(k = 1, \cdots, q)$. We call J the canonical map of M into A.

If $\Phi: \to B$ is a holomorphic map of M into a complex torus B, then there exists a homomorphism $f: A \to B$ such that

$$\Phi = T_{\Phi(x_0)} \circ f \circ J .$$

Assume now that M is compact and connected, and let us denote by $\mathfrak{h}$ the complex vector space of all holomorphic 1-forms on M. We may of course identify $\mathfrak{h}$ with the vector space Γ $(T^*(M))$ of holomorphic sections of the holomorphic cotangent bundle $T^*(M)$. Let e_x $(x \in M)$ be the linear map $\mathfrak{h} \to T^*_x(M)$ defined by $\omega \to \omega(x)$ for $\omega \in \mathfrak{h}$. We say that $T^*(M)$ is *ample* if e_x is surjective for all $x \in M$.

Assume now that there exists a holomorphic map Φ of M into a complex torus B, and let V be the subspace of $\mathfrak{h}$ spanned by $\omega^1, \cdots, \omega^m$, where ω^k are defined by (1.1). By (1.2) the differential $\Phi_*(x)$ of Φ_* at x is injective if and only if $e_x V = T^*_x(M)$. Therefore, if Φ is an immersion, then $e_x V = T^*_x(M)$ for every x. This shows that if M admits a holomorphic immersion into a complex torus, then $T^*(M)$ is ample. Conversely, if $T^*(M)$ is ample and M is Kählerian, then the canonical map J is a holomorphic immersion of M into the Albanese variety A.

Lemma 1. *Let M be a compact connected n-dimensional complex manifold. Assume that M admits a holomorphic immersion Φ into a complex torus B. Then the group* $\mathrm{Aut}_0(M)$ *is a complex torus and acts freely on M. Moreover, the kernel of the homomorphism Φ'*: $\mathrm{Aut}_0(M) \to B$ *is finite.*

Proof. The homomorphism $\Phi'_*: \mathfrak{G} \to C^m$ is injective. In fact, if $\xi \in \mathfrak{G}$ and $\Phi'_*(\xi) = 0$, then $\omega^k(\xi) = 0$ $(k = 1, \cdots, m)$ by (1.4), so that $\xi = 0$ because Φ is an immersion. Let $\xi \in \mathfrak{G}$, $\xi \neq 0$, and $\Phi'_*(\xi) = a$, and let h_t be the 1-parameter group of holomorphic transformations of M generated by ξ. Then $\Phi(h_t) = T_{\pi(ta)}$ for all $t \in R$, and the 1-parameter group of translations $T_{\pi(ta)}$ leaves the image $\Phi(M)$ invariant. Let $g_0 = \zeta^1\bar{\zeta}^1 + \cdots + \zeta^m\bar{\zeta}^m$ be the flat Kähler metric in the torus B, and $g = \Phi^* g_0$ the pullback of g_0 by Φ. Then g is a Kähler metric of M. We show that h_t is an isometry for all $t \in R$. Let y_0 be an arbitrary point of M, and U a neighborhood of y_0 such that Φ maps U biholomorphically onto a submanifold $\Phi(U)$ of B. Then $a = \Phi'_*(\xi)$ is tangent to $\Phi(U)$. Moreover, $T_{\pi(ta)}$ is an isometry of B (with respect to g_0) for each t, and leaves $\Phi(U)$ invariant provided $|t|$ is sufficiently small. It follows that a is an infinitesimal isometry (Killing vector field) of $\Phi(U)$, and therefore that the restriction of ξ to U is also an infinitesimal isometry in U with respect to g. Thus ξ is an infinitesimal isometry of the Kähler manifold M, and h_t is an isometry of M for each t. Since it is known that every infinitesimal isometry of a compact

Kähler manifold is a holomorphic vector field (cf. [2]), the group $\mathrm{Aut}_0\,(M)$ coincides with the identity component of the group of all isometries of M and is therefore compact. Since the kernel of $\Phi'_*: \mathfrak{G} \to C^m$ is trivial and $\mathrm{Aut}_0\,(M)$ is compact, the kernel of the homomorphism $\Phi': \mathrm{Aut}_0\,(M) \to B$ is finite. Now let $g \in \mathrm{Aut}_0\,(M)$ and $gy_0 = y_0$ for some $y_0 \in M$. Then $\Phi(y_0) = \Phi'(g)\Phi(y_0)$ and $\Phi'(g) = e$, the identity element of B. Hence $\Phi(gx) = \Phi(x)$ for any $x \in M$, and $\Phi_*(x_0) \circ g_*(x_0) = \Phi_*(x_0)$, where $g_*(x_0)$ denotes the differential of g at x_0. Since $\Phi_*(x_0)$ is injective, $g_*(x_0)$ should be the identity map. On the other hand, g is an isometry of M and maps a geodesic σ starting at x_0 with direction u to a geodesic $g \cdot \sigma$ starting at gx_0 with direction $g_*(x_0)u$. Since $g(x_0) = x_0$ and $g_*(x_0)u = u$ for any u, g leaves invariant any such geodesic pointwise. It follows then that g is the identity map in a neighborhood of x_0. Since this holds for any fixed point x_0 of g, we can conclude that g is the identity map, so that $\mathrm{Aut}_0\,(M)$ acts freely on M. Since $\mathrm{Aut}_0\,(M)$ is complex and compact, it is a complex torus.

Lemma 2. *Under the assumption in Lemma* 1, *let Γ be the closed subgroup of B consisting of all $b \in B$ such that $T_bX = X$, where $X = \Phi(M)$ is the image of M. Then the identity component Γ_0 of Γ coincides with the image $C = \Phi'(\mathrm{Aut}_0\,(M))$ of* $\mathrm{Aut}_0\,(M)$.

Proof. Clearly Γ is a closed subgroup of B and hence a closed Lie subgroup of B. Let b_t a 1-parameter subgroup of Γ. Then there exists a vector $a \in C^m$ such that $b_t = \pi(\mathrm{ta})$. Let $y_0 \in M$, and let U be a neighborhood of y_0 such that Φ maps U biholomorphically onto a submanifold $\Phi(U)$ of B. As in the proof of Lemma 1, we see that a is tangent to $\Phi(U)$ and is an infinitesimal isometry of $\Phi(U)$ with respect the Kähler metric of $\Phi(U)$ induced by the flat Kähler metric g_0 of B. Thus the vector field ξ_U on U corresponding to the restriction of a to $\Phi(U)$ is an infinitesimal isometry of M defined on U. Now let $a_k = \xi^k(a)$ and $\theta_a = \sum_{k=1}^m a_k \xi^k$. Then θ_a is a (0,1)-form on B such that $\theta_a(\bar{v}) = g_0(a, \bar{v})$ for any $v \in C^m$, so that $\Phi^*\theta_a$ is a (0,1)-form on M and there exists a unique vector field ξ of type (1,0) such that $g(\xi, \bar{\eta}) = (\Phi^*\theta_a)(\bar{\eta})$ for any vector field η of type (1,0) on M. We show that $\xi = \xi_U$ on U. Let $x \in U$. Then $\Phi_*(x)\xi_U = a$ and $g(\xi_U(x), \bar{\eta}(x)) = g_0(a, \Phi_*(x)\bar{\eta}(x)) = \theta_a(\Phi_*(x)\bar{\eta}(x)) = (\Phi^*\theta_a)(\bar{\eta}(x))$. Therefore $\xi(x) = \xi_U(x)$ and $\xi = \xi_U$ on U, which implies that ξ is an infinitesimal isometry on U. Since U is a neighborhood of an arbitrarily chosen point $y_0 \in M$, ξ is an infinitesimal isometry on M and hence holomorphic. Moreover, $\Phi_*(x)\xi(x) = a$ at each point as the above proof shows. Then $\Phi'(\xi) = a$, from which it follows that the one-parameter subgroup $b_t = \pi(\mathrm{ta})$ is contained in $C = \Phi'\,(\mathrm{Aut}_0\,(M))$ so that $L_0 \subset C$. Since clearly $C \subset \Gamma_0$, $\Gamma_0 = C$. q.e.d.

Since $\mathrm{Aut}_0\,(M)$ acts freely on M, the quotient $N = M/\mathrm{Aut}_0\,(M)$ is also a compact connected complex manifold. The image $C = \Phi'(\mathrm{Aut}_0\,(M))$ in B is a compact subgroup of B and moreover, since Φ' is a complex Lie group homomorphism, C is a complex subgroup and hence C is a complex subtorus of B.

The map Φ induces a holomorphic map Ψ of N into $B' = B/C$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\Phi} & B \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho'} \\ N & \xrightarrow[\Psi]{} & B' \end{array} \tag{1.5}$$

is commutative, where ρ and ρ' denote the canonical projection of M and B onto N and B' respectively.

It is easily seen that Ψ is a holomorphic immersion of N into B'. We show that the group $\mathrm{Aut}_0\,(N)$ consists of only the identity element. To see this let Γ'_0 be the identity component of the group of all $b' \in B'$ such that $T_{b'}X' = X'$, where $X' = \Psi(N)$. Let $b'(t)$ be a path in Γ'_0 such that $b'(0) = e'$, e' being the identity element of B', and let $b(t)$ be a path in B such that $\rho'(b(t)) = b'(t)$ and $b(0) = e$. Let x be an arbitrary point of M, and let $x' = \rho(x)$. Then $T_{b'(t)}\Psi(x') \in X'$, and hence there exists an element $y' \in N$ such that $T_{b'(t)}\Psi(x') = \Psi(y')$, where the parameter t is fixed. Let y be an element of M such that $\rho(x) = y'$. Then $T_{\rho'(b(t))}\Psi(\rho(x)) = \Psi(\rho(y))$, and from the commutativity of the diagram (1.5) we get $\rho'(\Phi(y)) = \rho'(T_{b(t)}\Phi(x))$. Thus there exists an element $c \in C$ such that $\Phi(y) = T_c(T_{b(t)}\Phi(x)) = T_{cb(t)}\Phi(x)$. This shows that $T_{cb(t)}X = X$, where $X = \Phi(M)$, which implies that $T_{b(t)}X = T_{c^{-1}}X = X$, so that $b(t) \in \Gamma$ in the notation of Lemma 2 for each t and hence that $b(t) \in \Gamma_0 = C$. Then $\rho'(b(t)) = e'$, and $b'(t) = e'$ for all t, which proves that Γ'_0 reduces to the identity element. Thus by Lemma 2 (applied for N and Ψ) we see that $\mathrm{Aut}_0\,(N)$ is trivial, and hence the following proposition.

Proposition 1. *Let M be a compact connected complex manifold, and assume that M admits a holomorphic immersion Φ into a complex torus B. Let* $\mathrm{Aut}_0\,(M)$ *be the identity component of the complex Lie group of all holomorphic transformations of M. Then* $\mathrm{Aut}_0\,(M)$ *is a complex torus acting freely on M. Let $N = M/\mathrm{Aut}_0\,(M)$ be the quotient of M by the free action of* $\mathrm{Aut}_0\,(M)$. *Then N is a compact connected complex manifold, N admits also a holomorphic immersion in a complex torus,* $\mathrm{Aut}_0\,(N)$ *is trivial, and the manifold M is a holomorphic principal fibre bundle over N of the structure group* $\mathrm{Aut}_0\,(M)$.

2. In the following sections we always denote by M a compact connected complex n-dimensional manifold, and assume that M admits a holomorphic immersion Φ into an $(n+1)$-dimensional complex torus B.

Let $X = \Phi(M)$. Then X defines a positive irreducible divisor $D(X)$ of B. More precisely there are an open covering $\{U_\alpha\}_{\alpha\in A}$ of B and holomorphic functions $\{f_\alpha\}_{\alpha\in A}$, each f_α being defined on U_α, such that f_α/f_β is holomorphic and nonvanishing on the intersection $U_\alpha \cap U_\beta$ and $X \cap U_\alpha$ is defined by the equation $f_\alpha = 0$ for each α. We can define $\{f_\alpha\}$ in the following way. Let $p \in B$. If $p \notin X$, then choose a neighborhood $U(p)$ of p such that $U(p) \cap X = \emptyset$, and define $f^{(p)} \equiv 1$. If $p \in X$, each preimage of p in M has a neighborhood which is mapped biholomorphically onto an n-dimensional submanifold of B passing through p, and we obtain a finite number of distinct n-dimensional submanifolds $X_1, \cdots, X_k$ each of which passes through p and is defined in a neighborhood $U(p)$ of p by an equation $f_j = 0$ such that $(df_j)(p) \neq 0$, where f_j is holomorphic in $U(p)$. Then define $f^{(p)} = f_1, \cdots, f_k$. The simple point of X is the point $p \in X$ for which $k = 1$. It is not difficult to check that the open covering $\{U(p)\}_{p \in B}$ and the holomorphic functions $\{f^{(p)}\}_{p \in B}$ verify the properties mentioned above, the set of simple points of X is a connected n-dimentional submanifold of B, and, for a simple point p, X is defined in a neighborhood of p by a single equation $f = 0$ such that $(df)(p) \neq 0$. This means that the positive divisor $D(X)$ is irreducible (cf. Weil [9, Appendix]). In the above notation put $g_{\alpha\beta} = f_\alpha / f_\beta$. Then $g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$, and $\{g_{\alpha\beta}\}$ is a system of transition functions of a holomorphic line bundle which we shall denote by $\{X\}$.

We recall here several facts about complex tori and theta functions (cf. Weil [9]). We shall write our complex torus B in the form $B = C^m/\Delta$ with $m = n + 1$ and Δ a lattice of C^m, and regard C^m as R^{2m} with complex structure J. Let $H = (h_{kj})$ be an $m \times m$ Hermitian matrix, and $H(u, v) = \sum h_{kj} u^k \bar{v}^j$ the corresponding Hermitian form. The imaginary part $A(u, v)$ of $H(u, v)$ is a skew symmetric bilinear form defined on $R^{2m} \times R^{2m}$ such that $A(Ju, Jv) = A(n, v)$. We say that A is *intgral* if A takes integral values on $\Delta \times \Delta$. If A is integral, there exists a basis $\{u_1, u_1', \cdots, u_m, u_m'\}$ of Δ such that

$$A(x, y) = \sum_{k=1}^{m} e_k (x^k y'^k - x'^k y^k) ,$$

where $x = \sum x^k u_k + \sum x'^k u_k'$, $y = \sum y^k u_k + \sum y'^k u_k'$, and the nonnegative integers e_k satisfy $e_1 | e_2 | \cdots | e_m$ (it is possible that $e_{p+1} = \cdots = e_m = 0$ for some $1 \leq p \leq m$). We call $e_1, \cdots, e_m$ the elementary divisors of A or of H. They are determined uniquely by A and hence by H.

A holomorphic function θ on C_m is called a theta function of type (H, Ψ) if

$$\theta(z + \sigma) = j(z, \sigma)\theta(z) \tag{2.1}$$

for all $z \in C^m$ and $\sigma \in \Delta$, where the automorphic factor $j(z, \sigma)$ is of the form

$$j(z, \sigma) = \Psi(\sigma) e\left[\frac{1}{2i} H(z, \sigma) + \frac{1}{4i} H(\sigma, \sigma)\right] , \tag{2.2}$$

where $e = \exp 2\pi i$, $H(u, v)$ is a Hermitian form on $C^m \times C^m$ whose imaginary part $A(u, v)$ is integral, and Ψ is a map of Δ into C^* satisfying $\Psi(\sigma + \sigma') \cdot e(\frac{1}{2} A(\sigma, \sigma')) = \Psi(\sigma)\Psi(\sigma')$ for any σ and σ' in Δ. We call an "automorphic factor" of the form (2.2) a theta factor of type (H, Ψ). A holomorphic theta function θ on C^m determines a divisor on B in the following way. Let $\{U_\alpha\}_{\alpha\in A}$ be an open covering of B such that each connected component of $\pi^{-1}(U_\alpha)$ ($\pi: C^m \to B$) is mapped homeomorphically onto U_α, and let $\tilde{U}_\alpha$ be one of the connected components of $\pi^{-1}(U_\alpha)$. Then $\pi^{-1}(U_\alpha) = \bigcup_{\sigma\in\Delta}(\sigma + \tilde{U}_\alpha)$. Let $\rho_\alpha = (\pi|\tilde{U}_\alpha)^{-1}$. Then $\{\theta_\alpha\}_{\alpha\in A}$, where $\theta_\alpha = \theta\circ\rho_\alpha$ for each $\alpha \in A$, defines a positive divisor D on B, which we shall denote by (θ) and is called the divisor of the theta function θ. Conversely, every positive divisor D is defind by a holomorphic theta function θ of type (H, Ψ) for some (H, Ψ), and the Chern class of the holomorphic line bundle $\{D\}$ is represented by the invariant (1,1)-form h_D of the form

$$h_D = \tfrac{1}{2}i \sum h_{kj}\zeta^k \wedge \bar{\zeta}^j ,$$

where $H = (h_{kj})$, and $\zeta^1, \cdots, \zeta^m$ denote, as in § 1, holomorphic 1-forms on B such that $\pi^*\zeta^k = dw^k$, $\{w^1, \cdots, w^m\}$ being the standard coordinates in C^m. It is known that if D is a positive divisor, then H is positive, and hence $c(\{D\}) \geq 0$ (see [9, Prop. 5, Chap. IV, No. 5]).

Let $b \in B$, and D be a positive divisor. Then the divisor T_bD is defined in an obvious way. Let Γ be the group of all $b \in B$ such that $T_bD = D$. Then the null space E_0 of the Hermitian form H associated to D has the following properties: Since the intersection $\Delta_0 = E_0 \cap \Delta$ is a lattice of E_0, the image $\pi(E_0)$ of E_0 in B is a complex subtorus isomorphic to E_0/Δ_0 and $\pi(E_0)$ is a subgroup of finite index in the group Γ (see Weil [9, Cor. 3, Chap. IV, No. 5]).

Let us apply these results to our divisor $D(X)$ defined by $X = \Phi(M)$. The Chern class $c(\{X\})$ of the line bundle $\{X\}$ is represented by a (1,1)-form h_Φ of the form

$$h_\Phi = \tfrac{1}{2}i \sum h_{kj}\zeta^k \wedge \bar{\zeta}^j ,$$

and we shall denote by H_Φ the Hermitian matrix (h_{kj}) and also by the same letter the associated Hermitian form on $C^m \times C^m$ $(m = n + 1)$. The null space of H_Φ will be denoted by E_0. Then $\pi(E_0)$ is a complex subtorus of B and is a subgroup of finite index of the group of all $b \in B$ such that $T_bD(X) = D(X)$. If $T_bD(X) = D(X)$, then we have $T_bX = X$, because $X = |D(X)|$, where $|D|$ denotes the carrier of a divisor D. Conversely, let $T_bX = X$. Clearly $|T_bD(X)| = T_b|D(X)| = |D(X)|$. The divisor $T_bD(X)$ is positive and irreducible together with $D(X)$, and they have the same carrier. Then $D(X)$ and $T_bD(X)$ coincide (cf. for example [9, Appendix].) Therefore $\pi(E_0)$ is a subgroup of finite index of the group Γ of all $b \in B$ such that $T_bX = X$. This combined with Lemma 2, § 1 gives $\Phi'(\mathrm{Aut}_0(M)) = \pi(E_0)$. Since Φ' and π are both local isomorphisms (cf. Lemma 1, § 1), we obtain

Theorem 1. *Let $\Phi: M \to B$ be a holomorphic immersion of an n-dimensional compact connected complex manifold M into an $(n + 1)$-dimensional* *complex torus B. Then the complex dimension of the complex torus* $\mathrm{Aut}_0(M)$ *is equal to the nullity of the Hermitian form H_Φ on $C^{n+1} \times C^{n+1}$ associated to Φ. Moreover, the Chern class $c(\{X\})$ of the line bundle $\{X\}$ $(X = \Phi(M))$ is represented by the* (1,1)*-form*

$$h_\Phi = \tfrac{1}{2} i \sum h_{kj} \zeta^k \wedge \bar{\zeta}^j ,$$

where $H_\Phi = (h_{kj})$, and H_Φ and hence $c(\{X\})$ are positive definite if and only if $\mathrm{Aut}_0(M)$ *is trivial.*

3. Let A be the imaginary part of the Hermitian form H_Φ. Then A is integral, and let $(e_1, \cdots, e_{n+1})$ be the elementary divisor of A. Let S be the singular set of X. Then $\Phi\colon M - \Phi^{-1}(S) \to X - S$ is a proper holomorphic map of the n-dimensional complex manifold $M - \Phi^{-1}(S)$ onto the n-dimensional complex manifold $X - S$, and for any $2n$-form η of compact carrier defined on $X - S$ we have

$$\int_{M-\Phi^{-1}(S)} \Phi^*\eta = d_\Phi \int_{X-S} \eta , \tag{3.1}$$

where d_Φ denotes the degree of $\Phi\colon M - \Phi^{-1}(S) \to X - S$ (see Sternberg [8]). On the other hand, X may be regarded as a $2n$-cycle in B, and the integral

$$\int_X \Psi$$

over X of an $2n$-form Ψ on B is defined. It follows from the definition of the integral over X (see Lelong [5]) and from (3.1) that

$$\int_M \Phi^*\Psi = d_\Phi \int_X \Psi \tag{3.2}$$

for any $2n$-form Ψ on B.

On the other hand, $c(\{X\})$ is the Poincaré dual of the homology class of X, that is,

$$\int_X \Psi = \int_B h_\Phi \wedge \Psi \tag{3.3}$$

holds for any $2n$-form Ψ on B (see Kodaira-Spencer [4, a] and Hirzebruch [1]). Letting $\Psi = h_\Phi^n$ in (3.2) and (3.3) we obtain

$$\int_M (\Phi^* h_\Phi)^n = d_\Phi \int_B h_\Phi^{n+1} . \tag{3.4}$$

Put $X' = X - S$ and $M' = M - \Phi^{-1}(S)$, and let $i_{X'}$ and $i_{M'}$ be the inclusion maps of X' and M' into B and M respectively. We also denote by Φ' the map $M' \to X'$ induced by Φ. Then $\Phi \circ i_{M'} = i_{X'} \circ \Phi'$.

The line bundle $\Phi'^*(i^*_{X'}\{X\})$ over M' is the restriction of the normal bundle N of M with respect to the holomorphic immersion Φ of M into B, and hence $i^*_{M'}N = \Phi'^*(i^*_{X'}\{X\})$, and $i^*_{M'}(\Phi^*h_\Phi)^n$ represents $i^*_{M'}c(N)^n$. Since $\Phi^{-1}(S)$ is an analytic subset of M and $M' = M - \Phi^{-1}(S)$, we have

$$(3.5) \qquad \int_M (\Phi^*h_\Phi)^n = \int_{M'} i^*_{M'}(\Phi^*h_\Phi)^n = \int_{M'} i^*_{M'}c(N)^n = \int_M c(N)^n .$$

We have an exact sequence

$$0 \to T(M) \to \Phi^*T(B) \to N \to 0 ,$$

and $\Phi^*T(B)$ is a trivial vector bundle of fibre dimension $n + 1$ since B is a complex torus. Thus we have

$$(1 + c_1(M) + \cdots + c_n(M))(1 + c(N)) = 1 ,$$

from which it follows that

$$(3.6) \qquad c_k(M) = (-1)^k c(N) \qquad (k = 1, 2, \cdots, n) ,$$

where $c_k(M)$ denotes the k-th Chern class of M. In particular $c(N)^n = (-1)^n c_n(M)$, and we have

$$\int_M c(N)^n = (-1)^n E(M) ,$$

where $E(M)$ denotes the Euler number of M. It follows then from (3.4) and (3.5) that

$$(3.7) \qquad E(M) = (-1)^n d_\Phi \int_B h_\Phi^{n+1} .$$

To compute the integral on the right hand side of (3.7), we notice

$$(3.8) \qquad \int_B h_\Phi^{n+1} = \int_F (\pi^*h_\Phi)^{n+1} ,$$

where F is a fundamental domain of Δ and $\pi^*h_\Phi = \frac{1}{2}i \sum h_{k\bar{j}} dw^k \wedge d\bar{w}^j$. Moreover,

$$H_\Phi(u, v) = A(Ju, v) + iA(u, v) ,$$

where J is the complex structure in $\boldsymbol{R}^{2(n+1)}$ defining $\boldsymbol{C}^{n+1}$, the real skew-symmetric form A may be regarded as a 2-form on $\boldsymbol{R}^{2(n+1)}$, and

$$(3.9) \qquad \pi^*h_\Phi = -A$$

as 2-forms on $R^{2(n+1)}$. Let $\{u_1, u'_1, \cdots, u_{n+1}, u'_{n+1}\}$ be a basis of Δ such that $A(u_k, u'_j) = e_k\delta_{kj}$ and $A(u_k, u_j) = A(u'_k, u'_j) = 0$ $(k, j = 1, 2, \cdots, n+1)$, and let $\{x^1, x'^1, \cdots, x^{n+1}, x'^{n+1}\}$ be the corresponding coordinates in $R^{2(n+1)}$. Then

$$A = \sum_k e_k dx^k \wedge dx'^k ,$$

$$\frac{1}{(n+1)!} A^{n+1} = (e_1 \cdots e_{n+1}) dx^1 \wedge dx'^1 \wedge \cdots \wedge dx^{n+1} \wedge dx'^{n+1} .$$

Since a fundamental domain F of Δ is given by

$$F = \{(x^1, x'^1, \cdots, x^{n+1}, x'^{n+1}) | 0 \leq x^k, x'^k \leq 1, k - 1, 2, \cdots, n+1\} ,$$

we get

$$\int_F A^{n+1} = \pm (n+1)! \, (e_1 \cdots e_{n+1}) , \tag{3.10}$$

where the sign depends on the orientation of the coordinates $\{u_1, u'_1, \cdots, u_{n+1}, u'_{n+1}\}$ of $R^{2(n+1)}$. It follows from (3.8), (3.9) and (3.10) that the absolute value of the integral $\int_B h_\Phi^{n+1}$ is equal to $(n+1)! \, (e_1 \cdots e_{n+1})$. However, the matrix H_Φ is positive so that $2(n+1)$-form h_Φ^{n+1} should be nonnegative. Thus the value of the integral should also be nonnegative, and we get

$$\int_B h_\Phi^{n+1} = (n+1)! \, (e_1 \cdots e_{n+1}) .$$

From (3.7) we obtain the formula

$$E(M) = (-1)^n (n+1)! \, d_\Phi (e_1 \cdots e_{n+1}) , \tag{3.11}$$

where the positive integer d_Φ is the degree of the map $\Phi: M - \Phi^{-1}(S) \to X - S$, and $(e_1, \cdots, e_{n+1})$ are elementary divisors of the imaginary part A of the Hermitian form H_Φ. Since the elementary divisors are all positive if and only if A and hence H_Φ are nondegenerate, we obtain

Theorem 2. *Let $\Phi: M \to B$ be a holomorphic immersion of an n-dimensional compact connected complex manifold M into an $(n+1)$-dimensional complex torus B. Then the following three conditions are equivalent:*

1) $\mathrm{Aut}_0 M = \{1\}$,

2) *H_Φ is positive definite, or, equivalently, the Chern class $c(\{X\})$ of the holomorphic line bundle $\{X\}$ over B is positive definite,*

3) *the Euler number $E(M)$ of M is not zero.*

Since the pullback of $c(\{X\})$ by Φ defines a Hodge form on M, we obtain from Kodaira's theorem the following corollary which is a special case of a more general theorem proved in [6].

Corollary. *Let M be an n-dimensional compact connected complex manifold admitting a holomorphic immersion into an $(n+1)$-dimensional complex torus. If $E(M) \neq 0$, then M is algebraic.*

Let us consider now the quotient $N = M/\mathrm{Aut}_0(M)$. Then by § 1, N admits a holomorphic immersion into $B' = B/C$, where $C = \Phi'(\mathrm{Aut}_0(M))$. However $\dim C = \dim \mathrm{Aut}_0(M)$ by Lemma 1, and hence $\dim B' = \dim N + 1$. Since $\mathrm{Aut}_0(N) = \{1\}$ by Proposition 1, from Theorem 2 and its corollary we get

Theorem 3. *Let M be an n-dimensional compact connected complex manifold admitting a holomorphic immersion in an $(n+1)$-dimensional complex torus B. Then M is a principal fibre bundle over the quotient manifold $N = M/\mathrm{Aut}_0(M)$ of the structure group $\mathrm{Aut}_0(M)$, and N has the following properties:*

a) *N is algebraic, and N admits a holomorphic immersion Ψ into a complex torus B' with $\dim B' = \dim N + 1$.*

b) *$E(N) \neq 0$, and the Chern class $c(\{X'\})$ of $\{X'\}$ is positive definite, where $\{X'\}$ denotes the holomorphic line bundle defined by the divisor associated with the image $X' = \Psi(N)$ of N in B'.*

Remark. It follows from a recent result of Nagano-Smyth [7] that, without assuming that $\dim B = n+1$, $\mathrm{Aut}_0(M) = \{1\}$ if and only if the Ricci tensor of the Kähler metric on M, which is induced from a flat Kähler metric on B, is negative definite almost everywhere (it is always negative everywhere). Thus the Chern number c_1^n of M is nonzero, and N is algebraic by a theorem proved in [6].

4. We shall prove the following theorem.

Theorem 4. *Let M be an n-dimensional compact connected Kähler manifold. Assume that the Euler number $E(M)$ of M is not zero and $n \geq 2$. Then M admits a holomorphic immersion into an $(n+1)$-dimensional complex torus if and only if the following two conditions are satisfied:*

1) *the cotangent bundle $T^*(M)$ of M is ample,*

2) *$h^{1,0}(M) = n+1$, where $h^{1,0}(M)$ is the number of linearly independent holomorphic 1-forms on M.*

Proof. If the above two conditions are verified, then without assuming that $n \geq 2$ and $E(M) \neq 0$ we see that the canonical map J from M into the Albanese variety of M is a holomorphic immersion as we have already observed in § 1.

Suppose now that M has a holomorphic immersion Φ into a complex torus B of dimension $n+1$. Then $T^*(M)$ is ample (see § 1), and the condition 2) follows from the following lemma if we assume $E(M) \neq 0$ and $n \geq 2$.

Lemma 3. *Assume that $E(M) \neq 0$, and M admits a holomorphic immersion into a complex torus B of dimension $n+1$, where $n = \dim M$. Then we have*

$$(n+1)h^{0,q}(M) = h^{1,q}(M) \qquad (q = 0, 1, \cdots, n-2)$$

In particular, $h^{1,0}(M) = n + 1$ *for* $n \geq 2$.

Let N be the normal bundle of M with respect to the immersion $\Phi: M \to B$. Then we have

$$0 \to T(M) \to I_{n+1} \to N \to 0 ,$$

where $I_{n+1} = \Phi^*T(B)$ denotes the trivial vector bundle of fibre dimension $n + 1$. Then we get

$$0 \to N^* \to I_{n+1} \to T^*(M) \to 0$$

and hence the exact sequence of cohomologies:

$$\to H^q(M, N^*) \to H^q(M, \mathfrak{o}^{n+1}) \to H^q(M, T^*(M)) \to H^{q+1}(M, N^*) \to .$$

Let us consider the Chern class $c(N)$ of N. As we have seen in § 3, N and $\Phi^*\{X\}$ coincide on $M' = M - \Phi^{-1}(S)$. Let $F = N \cdot \Phi^*\{X\}^{-1}$, and for each small $\varepsilon > 0$ let U_ε be an open neighborhood of $\Phi^{-1}(S)$ such that the measure (with respect to any Kähler metric of M) of U_ε tends to 0 as ε tends to 0. For each ε, $c(F)$ is represented by a (1,1)-form η_ε which is 0 outside U_ε. Since $N = \Phi^*\{X\} \cdot F$, $c(N)$ is represented by $\Phi^*h_\Phi + \eta_\varepsilon$. Since $E(M) \neq 0$, h_Φ and hence Φ^*h_Φ are positive definite by Theorem 2. Under this situation Kodaira's original proof of his vanishing theorem (see Kodaira [3]) works well, and we can conclude that $H^q(M, N^*) = 0$ for $q < n$. It thus follows from the above exact sequence of cohomologies that $H^q(M, \mathfrak{o}^{n+1}) \cong H^q(M, T^*(M))$ for $q + 1 < n$, and this proves our lemma and at the same time Theorem 4.

Remark. The assumption $n \geq 2$ is necessary. In fact, we can immerse a compact Riemann surface M of genus $g > 2$ for some g into a 2-dimensional complex torus whereas $h^{1,0}(M) = g > 2$.

Under the assumption in Theorem 4 we show that Φ induces an isomorphism of $H^{1,0}(B)$ onto $H^{1,0}(M)$. We know by Lemma 3 that $\dim H^{1,0}(M) = n + 1$, and $H^{1,0}(B)$ is spanned by $\zeta^1, \cdots, \zeta^{n+1}$, so that $\dim H^{1,0}(B) = n + 1$. Therefore it is sufficient to show that $\Phi^*: H^{1,0}(B) \to H^{1,0}(M)$ is injective. By a linear change of coordinates in C^{n+1}, we may assume that the kernel of $\Phi^*: H^{1,0}(B) \to H^{1,0}(M)$ is spanned by $\zeta^1, \cdots, \zeta^s$ $(s \geq 0)$. As in § 1, let $\omega^k = \Phi^*\zeta^k$. Since $\omega^1 = \cdots = \omega^s = 0$, $\omega^{s+1}, \cdots, \omega^{n+1}$ should span the cotangent vector space of M at each point of M. Therefore $s \leq 1$. If $s = 1$, then $\omega^2, \cdots, \omega^{n+1}$ must be linearly independent at each point of M, and M is parallizable, which contradicts the assumption that $E(M)$ is not zero. Let now $B = C^{n+1}/\Delta_B$, and $A = C^{n+1}/\Delta_A$ where A denotes the Albanese variety of M. We denote by π_B and π_A the projections of C^{n+1} onto B and A respectively. Then by § 1

$$\Phi(x) = T_{\Phi(x_0)}\left(\pi_B\left(\int_{x_0}^x \omega^1, \cdots, \int_{x_0}^x \omega^{n+1}\right)\right) ,$$

$$J(x) = \pi_A\left(\int_{x_0}^{x} \omega^1, \cdots, \int_{x_0}^{x} \omega^{n+1}\right).$$

For any closed path γ starting at x_0 we have

$$\pi_B\left(\int_\gamma \omega^1, \cdots, \int_\gamma \omega^{n+1}\right) = \text{the identity element } e_B \text{ of } B ,$$

and

$$\left(\int_\gamma \omega^1, \cdots, \int_\gamma \omega^{n+1}\right) \in \Delta_B ,$$

which shows that $\Delta_A \subset \Delta_B$. Therefore there is a surjective homomorphism $f: A \to B$ such that $f\left(\pi_A\left(\int_{x_0}^{x} \omega^1, \cdots, \int_{x_0}^{x} \omega^{n+1}\right)\right) = \pi_B\left(\int_{x_0}^{x} \omega^1, \cdots, \int_{x_0}^{x} \omega^{n+1}\right)$. Thus we have shown that for any holomorphic immersion Φ of M into an $(n+1)$-dimensional torus B there exists a homomorphism f of the Albanese variety A of M onto B such that

$$\Phi(x) = T_{\Phi(x_0)}(f(J(x)))$$

for all $x \in M$.

Let $H^{1,1}(M, \boldsymbol{R})$ and $H^{1,1}(B, \boldsymbol{R})$ denote respectively the subspaces of the real de Rham cohomology group $H^2(M, \boldsymbol{R})$ and $H^2(B, \boldsymbol{R})$ whose elements are represented by closed real forms of type (1,1). We show that $\Phi^*: H^{1,1}(B, \boldsymbol{R}) \to H^{1,1}(M, \boldsymbol{R})$ is bijective provided $n \geq 3$. Since $\dim_{\boldsymbol{R}} H^{1,1}(B, \boldsymbol{R}) = h^{1,1}(B) = (n+1)^2$ and $\dim_{\boldsymbol{R}} H^{1,1}(M, \boldsymbol{R}) = h^{1,1}(M)$, $h^{1,1}(M) = (n+1)h^{0,1}(M) = (n+1)^2$ by Lemma 3 provided $n \geq 3$. Therefore it is sufficient to show that Φ^* is injective. The space $H^{1,1}(B, \boldsymbol{R})$ is identified with the space of all (1,1)-forms on B of the form $\theta = i \sum \theta_{kj}\zeta^k \wedge \bar{\zeta}^k$ where $\Theta = (\theta_{kj})$ is a constant Hermitian matrix. Let $\omega^k = \Phi^*\zeta^k$ $(k = 1, \cdots, n+1)$, and suppose that $\Phi^*\theta \sim 0$. Then Θ cannot be (positive or negative) definite. In fact, if Θ is positive definite for example, then $\Phi^*\theta$ is the Kähler form of the Kähler metric $\sum_{k,j}\theta_{kj}\omega^k\bar{\omega}^j$ on M, and $\Phi^*\theta$ cannot be cohomologous to zero. After a suitable linear change of coordinates in $\boldsymbol{C}^{n+1}$, we may assume that Θ is of the form

$$\Theta = \begin{pmatrix} O_k & & \\ & E_j & \\ & & -E_l \end{pmatrix} \qquad (k + j + l = n + 1) ,$$

where O_k denotes the $k \times k$ zero matrix, and E_j and E_l are the unit matrices of types $j \times j$ and $l \times l$ respectively. We may assume $j \leq l$ (otherwise replace θ by $-\theta$). To simplify our notation, put $\zeta_k = \zeta^k \wedge \bar{\zeta}^k$, $\omega_k = \omega^k \wedge \bar{\omega}^k$, $\omega = i \sum \omega_k$. Then $\theta = i(\zeta_{k+1} + \cdots + \zeta_{k+j} - \zeta_{k+j+1} - \cdots - \zeta_{n+1})$, and $\omega + \Phi^*\theta = \omega'$ where $\omega' = i(\omega_1 + \cdots + \omega_k + 2\omega_{k+1} + \cdots + 2\omega_{k+j})$. Since $\Phi^*\theta \sim 0$,

$\omega \sim \omega'$ and hence $\omega^n \sim \omega'^n$. If $k + j < n$, then $\omega'^n = 0$ and $\omega^n \sim 0$, and this is impossible because ω is the Kähler form of the Kähler metric $\sum \omega^k \cdot \bar{\omega}^k$. Therefore $k + j = n$ or $k + j = n + 1$. Suppose $k + j = n$. Since $k + j + l = n + 1$, we get $l = 1$, and as we have assumed that $j \leq l$, j is either equal to 0 or 1. If $j = 0$, we have $k = n$, and

(A) $\quad \Phi^*\theta = -i\omega^{n+1} \wedge \bar{\omega}^{n+1} \sim 0\,, \qquad \omega \sim \omega' = i(\omega_1 + \cdots + \omega_n)\,.$

If $j = 1$, we have $k = n - 1$, $l = j = 1$ and

(B) $\quad \Phi^*\theta = i(\omega_n - \omega_{n+1}) \sim 0\,, \quad \omega \sim \omega' = i(\omega_1 + \cdots + \omega_{n-1} + 2\omega_n)\,.$

If $k + j = n + 1$, then $l = 0$, and hence $j = 0$ and $\Theta = 0$ which implies $\theta = 0$.

In either of the cases (A) and (B), we have $\omega^n \sim \omega'^n$ and

$$\int_M \omega^n = \int_M \omega'^n \tag{4.1}$$

with $\omega^n = \Phi^*(i \sum_{k=1}^{n+1} \zeta_k)^n$ and either $\omega' = \Phi^*(i \sum_{k=1}^{n} \zeta_k)^n$ (for case (A)) or $\omega'^n = \Phi^*(i \sum_{k=1}^{n-1} \zeta_k + 2i\zeta_n)^n$ (for case (B)). Put $\zeta = i \sum_{k=1}^{n+1} \zeta_k$ and $\zeta' = i \sum_{k=1}^{n} \zeta_k$ (for case (A)) or $\zeta' = i \sum_{k=1}^{n-1} \zeta_k + 2i\zeta_n$ (for case (B)). Then by (3.2) and (4.1)

$$\int_X \zeta^n = \int_X \zeta'^n\,,$$

where $X = \Phi(M)$. Since $c(\{X\})$ is the Poincaré dual of the homology class X, we get

$$\int_B h_\Phi \wedge \zeta^n = \int_B h_\Phi \wedge \zeta'^n$$

and hence $h_\Phi \wedge \zeta^n = h_\Phi \wedge \zeta'^n$ because both sides are invariant and of type (n, n). As in § 3, let $h_\Phi = \frac{1}{2} i \sum h_{kj} \zeta^k \wedge \bar{\zeta}^j$. Since $E(M) \neq 0$, the Hermitian matrix $H_\Phi = (h_{kj})$ is positive definite by Theorem 2, and $\zeta^n = i^n n! \sum_{k=1}^{n+1} \zeta_1 \wedge \cdots \wedge \hat{\zeta}_k \wedge \cdots \wedge \zeta_{n+1}$, $h_\Phi \wedge \zeta^n = \frac{1}{2} i^{n+1} n! \operatorname{Tr} H_\Phi(\zeta_1 \wedge \cdots \wedge \zeta_{n+1})$. Suppose that the case (A) occurs. Then $\zeta'^n = i^n n!\, \zeta_1 \wedge \cdots \wedge \zeta_n$, and $h_\Phi \wedge \zeta'^n = \frac{1}{2} i^{n+1} n!\; h_{n+1,n+1}(\zeta_1 \wedge \cdots \wedge \zeta_{n+1})$. It follows from $h_\Phi \wedge \zeta^n = h_\Phi \wedge \zeta'^n$ that $\sum_{k=1}^{n} h_{kk} = 0$. However $n \times n$ Hermitian matrix $H' = (h_{kj})$ $(1 \leq k, j \leq n)$ is positive definite, and it is impossible that $\operatorname{Tr} H' = 0$, so the case (A) cannot occur. Suppose now that the case (B) occurs. Then $\zeta' = i(\sum_{k=1}^{n-1} \zeta_k + 2\zeta_n)$, $\zeta'^n = i^n n!\; 2(\zeta_1 \wedge \cdots \wedge \zeta_n)$, and $h_\Phi \wedge \zeta'^n = i^{n+1} n!\; h_{n+1,n+1}(\zeta_1 \wedge \cdots \wedge \zeta_{n+1})$. It follows from $h_\Phi \wedge \zeta^n = h_\Phi \wedge \zeta'^n$ that $\operatorname{Tr} H_\Phi = 2h_{n+1,n+1}$. In the case (B), $\omega_n \sim \omega_{n+1}$, and since each ω_k is closed, we have also $\omega_1 \wedge \cdots \wedge \omega_{n-1} \wedge \omega_n \sim \omega_1 \wedge \cdots \wedge \omega_{n-1} \wedge \omega_{n+1}$ and hence

$$\int_M \omega_1 \wedge \cdots \wedge \omega_{n-1} \wedge \omega_n = \int_M \omega_1 \wedge \cdots \wedge \omega_{n-1} \wedge \omega_{n+1} .$$

From this we obtain as before $h_\Phi \wedge \zeta_1 \wedge \cdots \wedge \zeta_{n-1} \wedge \zeta_n = h_\Phi \wedge \zeta_1 \wedge \cdots \wedge \zeta_{n-1} \wedge \zeta_{n+1}$, and $h_{n+1,n+1}\zeta_1 \wedge \cdots \wedge \zeta_{n+1} = h_{n,n}\zeta_1 \wedge \cdots \wedge \zeta_{n+1}$ which implies $h_{n+1,n+1} = h_{n,n}$. Combining this with $\mathrm{Tr}\, H_\Phi = 2h_{n+1,n+1}$ we get $\sum_{k=1}^{n-1} h_{kk} = 0$, and this contradicts the fact that the $(n-1) \times (n-1)$ Hermitian matrix (h_{kj}) $(1 \leq k, j \leq n-2)$ is also positive definite. Thus both cases (A) and (B) cannot occur, and therefore θ should be equal to 0; this concludes our proof.

It is also shown that every cohomology class in $H^{1,1}(M, \boldsymbol{R})$ is represented by a unique (1,1)-form of the type

$$i \sum \theta_{kj} \omega^k \wedge \bar{\omega}^j ,$$

where (θ_{kj}) is a constant Hermitian matrix of type $(n+1) \times (n+1)$, and $\{\omega^1, \cdots, \omega^{n+1}\}$ with $\omega^k = \Phi^* \zeta^k$ is a basis of the space of holomorphic 1-forms on M. From this and a theorem of Weil it follows that every divisor on M is a divisor of a theta function on the universal covering manifold $\tilde{M}$ of M (see Weil [9, Th. 2 on P. 99]).

Summing up and combining with Theorem 4 we obtain

Theorem 5. *Let M be a compact connected Kähler manifold such that $T^*(M)$ is ample, $E(M) \neq 0$ and $h^{1,0}(M) = \dim M + 1$. Then the canonical map $J: M \to A$ of M into the Albanese variety is an immersion. Moreover, if $n = \dim M \geq 3$, then, J induces a bijection of $H^{1,1}(A, \boldsymbol{R})$ onto $H^{1,1}(M, \boldsymbol{R})$, every cohomology class of $H^{1,1}(M, \boldsymbol{R})$ is represented by a unique (1,1)-form of the type $\theta = i \sum \theta_{kj} \omega^k \wedge \bar{\omega}^j$, where $\{\omega^1, \cdots, \omega^{n+1}\}$ is a basis of the space of holomorphic 1-forms on M and (θ_{kj}) is a constant Hermitian matrix and every divisor of M is the divisor of a theta function on the universal covering manifold $\tilde{M}$ of M.*

5. In this section we always denote by M an n-dimensional compact connected Kähler manifold admitting a holomorphic immersion into an $(n+1)$-dimensional complex torus, and assume that $E(M) \neq 0$. Then the canonical map $J: M \to A$ of M into the Albanese variety A is an immersion, and A is $(n+1)$-dimensional. The Chern class $c(\{X\})$ of the line bundle $\{X\}$ associated to $X = J(M)$ is positive definite, and is represented by a unique (1, 1)-form

$$h_J = \tfrac{1}{2} i \sum h_{kj} \zeta^k \wedge \bar{\zeta}^j .$$

Here the constant Hermitian matrix H_J is positive definite, and the imaginary part A of the associated Hermitian form is integral valued on $\Delta_A \times \Delta_A$ where we write $A = \boldsymbol{C}^{n+1}/\Delta_A$. The elementary divisors of A (or H_J) denoted by $e_1, \cdots, e_{n+1}$ are all positive integers since A is nondegenerate. For any n-tuple of integers $\pi = (p_1, \cdots, p_n)$ such that $\sum k p_k = n$, let

$$c_\pi[M] = \int_M c_1^{p_1} \cdots c_n^{p_n} ,$$

where c_k denotes the k-th Chern class of M. These numbers are called the Chern numbers of M. By (3.6), $c_1^{p_1}\cdots c_n^{p_n} = (-1)^n c(N)^n$ for all π so that the Chern numbers are all equal. Since $E(M)$ is equal to $c_\pi[M]$ with $\pi = (0, 0\cdots, 1)$, every Chern number of M is equal to $E(M)$:

$$E(M) = c_\pi[M] \qquad \text{for any } \pi .$$

Let Y be another n-dimensional compact connected Kähler manifold with $E(Y) \neq 0$ admitting a holomorphic immersion in an $(n+1)$-dimensional complex torus. Then we also have $E(Y) = c_\pi[Y]$ for any π and hence

$$(5.1) \qquad c_\pi[M] = \rho\cdot c_\pi[Y] \qquad \text{for any } \pi ,$$

where $\rho = E(M)/E(Y)$ is a nonzero rational number.

We say that two n-dimensional compact connected Kähler manifolds M and Y are proportional if their Chern numbers satisfy (5.1) for some nonzero rational ρ (cf. Hirzebruch [1]). It follows then from Riemann-Roch-Hirzebruch theorem that

$$(5.2) \qquad \chi(M) = \rho\cdot\chi(Y) , \qquad \chi(M, K_M^r) = \rho\cdot\chi(Y, K_Y^r) ,$$

where K_M and K_Y denote the canonical bundles of M and Y respectively, and r is an integer. We shall compute $\chi(M)$ and $\chi(M, K_M^r)$ by choosing Y suitably.

Since $E(M) \neq 0$, M is algebraic (Theorem 2, Cor.) and hence A is algebraic. Let Y be a nonsingular hyperplane section of A (with respect to a projective imbedding of A). By Lefschetz theorem, $b_p(Y) = b_p(A) = \binom{2n+2}{p}$ for $p = 0, 1, \cdots, n-1$ and $b_n(Y) \geq b_n(A) = \binom{2n+2}{n}$. By a computation using the Poincaré dualty we see that

$$(5.3) \qquad E(Y) = (-1)^n\left\{b_n(Y) - b_n(A) + \frac{(2n+2)!}{(n+1)!\,(n+2)!}\right\} ,$$

so that $E(Y) \neq 0$, and the identity map of Y into A is an immersion (as a matter of fact, the identity map of Y into A is the canonical map, and A is the Albanese variety of Y).

We shall denote by $\{Y\}$ the line bundle over A associated with the nonsingular positive divisor Y of A. The Chern class $c(\{Y\})$ of $\{Y\}$ is represented by a unique $(1, 1)$-form of the type

$$\theta_Y = \tfrac{1}{2}i \sum \theta_{kj}\zeta^k \wedge \bar\zeta^j ,$$

where $H_Y = (\theta_{kj})$ is a constant positive definite Hermitian matrix, and the imaginary part of the associated Hermitian form is integral valued on $\Delta_A \times \Delta_A$.

We denote by $f_1, \cdots, f_{n+1}$ the elementary divisor of H_Y. As a special case of the formula (3.11) we get

$$E(Y) = (-1)^n (n+1)!\,(f_1 \cdots f_{n+1}) . \tag{5.4}$$

We also obtain from (5.3) and (5.4)

$$b_n(Y) = \frac{n(2n+2)!}{(n+1)!\,(n+2)!} + (n+1)!\,(f_1 \cdots f_{n+1}) .$$

We shall use the following two formulas:

1) Riemann-Roch-Hirzebruch formula for complex tori: For any holomorphic line bundle F over A

$$\chi(A, F) = \frac{1}{(n+1)!} \int_A c(F)^{n+1} . \tag{5.5}$$

(This formula follows easily from the Riemann-Roch-Hirzebruch theorem, because Chern classes of A are all zero.)

2) Kodaira-Spencer formula [4, b]: For any holomorphic line bundle F over A and for any nonsingular divisor Y of A

$$\chi(A, F) = \chi(A, F \otimes \{Y\}^{-1}) + \chi(Y, F|Y) . \tag{5.6}$$

Let F be the trivial line bundle over A. Then $\chi(A, F) = \chi(A) = \sum_q (-1)^q h^{0,q}(A) = \sum (-1)^q \binom{n}{q} = 0$. Hence $\chi(Y) = -\chi(A, \{Y\}^{-1})$ by (5.6), and

$$\chi(A, \{Y\}^{-1}) = \frac{(-1)^{n+1}}{(n+1)!} \int_A c(\{Y\})^{n+1} = (-1)^{n+1}(f_1 \cdots f_{n+1})$$

by (5.5) as in § 3, so that $\chi(Y) = (-1)^n (f_1 \cdots f_{n+1})$ which together with (5.4) gives

$$\chi(Y)/E(N) = 1/(n+1)! . \tag{5.7}$$

It follows from (5.2), (5.7) and (3.11) that

$$\chi(M) = \frac{E(M)}{E(Y)} \chi(Y) = \frac{E(M)}{(n+1)!} = (-1)^n d_J (e_1 \cdots e_{n+1}) ,$$

where $e_1, \cdots, e_{n+1}$ are the elementary divisors of H_J, and d_J is the degree of the map $J: M - J^{-1}(S) \to X - S$, S denoting the singularities of $X = J(M)$. On the other hand

$$\chi(Y) = \sum_{q=1}^{n} (-1)^b h^{0,q}(Y) = \sum_{q=0}^{n-1} (-1)^q \binom{n+1}{q} + (-1)^n h^{0,q}(Y)$$
$$= -(-1)^n \binom{n+1}{n} - (-1)^{n+1} + (-1)^n h^{0,n}(Y) .$$

We have shown that $\chi(Y) = (-1)^n (f_1 \cdots f_{n+1})$ so that

$$h^{0,n}(Y) = n + f_1 \cdots f_{n+1} .$$

Now putting $F = \{Y\}^r$ in (5.6) gives $\chi(A, \{Y\}^r) = \chi(A, \{Y\}^{r-1}) + \chi(Y, N^r)$, where $N = \{Y\}|Y$ is the normal bundle of Y. By (5.5) we get

$$\chi(Y, N^r) = \frac{1}{(n+1)!} \int_A (r^{n+1} - (r-1)^{n+1}) c(\{Y\})^{n+1}$$
$$= (r^{n+1} - (r-1)^{n+1}) f_1 \cdots f_{n+1} .$$

On the other hand $c(K_Y) = -c_1(Y) = c(N)$, and from the Riemann-Roch-Hirzebruch theorem for Y it follows that $\chi(Y, N^r) = \chi(Y, K_Y^r)$ so that $\chi(Y, K_Y^r) = (r^{n+1} - (r-1)^{n+1}) f_1 \cdots f_{n+1} = (-1)^n (r^{n+1} - (r-1)^{n+1}) E(Y)/(n+1)!$. Thus $\chi(Y, K_Y^r)/E(Y) = (-1)^n (r^{n+1} - (r-1)^{n+1})/(n+1)!$, and from $\chi(M, K_M^r) = (E(M)/E(Y))\chi(Y, K_Y^r)$ we obtain the following formula:

$$\chi(M, K_M^r) = d_J (r^{n+1} - (r-1)^{n+1})(e_1 \cdots e_{n+1}) .$$

We have $c(K_M^r \otimes K_M^{-1}) = (r-1)c(K_M) = -(r-1)c_1(M) = (r-1)c(N)$ by (3.6), where N is the normal bundle of M with respect to the immersion J of M into A. Let S be the singularities of $X = J(M)$ and, as in the proof of Lemma 3, let $U_\varepsilon(\varepsilon > 0)$ be a neighborhood of $J^{-1}(S)$ such that the measure of U_ε tends to zero as ε tends to zero. For each ε, $c(K_M^r \otimes K_M^{-1})$ is represented by $(r-1) J^* h_J + \eta_\varepsilon$, where $\eta_\varepsilon = 0$ outside U_ε. If $r > 1$, then $(r-1) J^* h_J$ is positive definite everywhere, and from this we conclude as indicated in the proof of Lemma 3 that $H^q(M, K_M^r) = 0$ for $q > 0$ provided $r > 1$. Therefore we get also

$$\dim H^0(M, K_M^r) = d_J (r^{n+1} - (r-1)^{n+1})(e_1 \cdots e_{n+1}) \qquad \text{for } r > 1 .$$

Remark. It can be proved that the Ricci tensor $R_{k\bar{j}}$ of the Kähler metric on M, which is the pullback of a flat Kähler metric in A, is negative everywhere, and is negative definite almost everywhere if $E(M) \neq 0$. Since $c(K_M)$ is represented by the (1,1)-form $-\frac{1}{2}(i/\pi)R_{k\bar{j}} dz^k \wedge d\bar{z}^j$, $c(K_M)$ is positive definite almost everywhere. From this fact we can also prove the vanishing of cohomology groups discussed above and also in the proof of Lemma 3.

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